Since $\Phi(t_{n+1},t_n + \Delta)$ is deterministic, it may be placed inside of the ensemble average operator and the integration sign. Also, since $\mathbf{w}_n$ and $\mathbf{y}_n$ are integrals of white noise over separate intervals their cross-correlation must be zero;

$$\langle \mathbf{y}_n\mathbf{w}_n{}^T\rangle = \langle \mathbf{w}_n\mathbf{y}_n{}^T\rangle = 0 \tag{14}$$

and

$$\langle \mathbf{y}_n\mathbf{y}_n{}^T\rangle + \langle \mathbf{w}_n\mathbf{w}_n{}^T\rangle = \langle(\mathbf{w}_n + \mathbf{y}_n)(\mathbf{w}_n + \mathbf{y}_n)^T\rangle \tag{15}$$

Using Eq. (15), and the fact that

$$\Phi(t_{n+1},t_n + \Delta)\Phi(t_n + \Delta,\tau) = \Phi(t_{n+1},\tau) \tag{16}$$

Eq. (13) becomes

$$\Phi(t_{n+1},t_n + \Delta)W_n\Phi^T(t_{n+1},t_n + \Delta) + Y_n = \left\langle \left(\int_{t_n}^{t_{n+1}} \Phi(t_{n+1},\tau)\mathbf{u}(\tau)d\tau\right)\left(\int_{t_n}^{t_{n+1}} \Phi(t_{n+1},\tau)\mathbf{u}(\tau)d\tau\right)^T \right\rangle \triangleq Q_n \tag{17}$$

and Eq. (12) becomes

$$P_{n+1}(-) = \Phi(t_{n+1},t_n)\{[I\text{-}AK_nH_n]P_n(-)[I\text{-}AK_nH_n]^T + AK_nR_nK_n{}^TA^T\}\Phi^T(t_{n+1},t_n) + Q_n \tag{18}$$

It can be seen that the preceding relation for the error covariance, Eq. (18), is identical to the one obtained when there is no time delay. Therefore, if $A\Phi^* = \Phi A$ and Eq. (1) is employed, the time delay has no effect on errors subsequent to the last update and correction, Δ sec after the final measurement. There is an unavoidable difference in errors over the interval between a measurement and the correction which follows it. The condition on the matrices A and Φ^* is satisfied if all states of the problem except those representing correlated measurement errors are corrected after each measurement.

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Finite Deflection of Elliptical Plates on Elastic Foundations

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Nomenclature

D	= flexural rigidity of plate
E	= modulus of elasticity of plate material
G	= shear modulus
K	= dimensionless foundation modulus
Q	= dimensionless uniformly distributed load
R	= aspect ratio, b/a
U,V,W	= dimensionless displacement parameters
W_0	= dimensionless lateral center deflection, w_0/h
$2a,2b$	= length of minor and major axis of elliptical plate, respectively
h	= thickness of plate
k	= elastic foundation reaction per unit area per unit deflection
q	= intensity of uniformly distributed load
u,v,w	= displacement along the x, y, and z axes, respectively

THE finite deflection behavior of thin elastic plates was first formulated mathematically by von Kármán in the form of two coupled nonlinear partial differential equations.[1]

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Approximate solutions for these equations have been obtained for the case of the rectangular plate by Way[2] using the Rayleigh-Ritz technique, and also by Levy and Greenman,[3] who substituted a double Fourier series solution into the differential equations and evaluated the coefficients. Invariably, the methods of analysis employed are extremely laborious and require considerable computations.

In this work, a simple and yet sufficiently accurate method based on the small parameter perturbation technique is used to analyze the small and finite deflection behavior of uniformly loaded clamped elliptical plates resting on an elastic foundation. This technique has been applied successfully to circular plates with no elastic support by Chien,[4] and, more recently, to a square plate by Chien and Yeh.[5]

Throughout the following analysis, the plate is considered to be elastic and the foundation is considered to be of the Winkler type; i.e., foundation reaction is proportional to the deflection.

For plates resting on elastic foundation and with moderately large deflections, the von Kármán equations can be slightly modified and three nonlinear partial differential equations governing the lateral deflection and in plane displacements can be written as

$$u_{,xx} + w_{,x}w_{,xx} + \nu(v_{,xy} + w_{,y}w_{,xy}) + \tfrac{1}{2}(1 - \nu)(u_{,yy} + v_{,xy} + w_{,x}w_{,yy} + w_{,y}w_{,xy}) = 0 \tag{1}$$

$$v_{,yy} + w_{,y}w_{,yy} + \nu(u_{,xy} + w_{,x}w_{,xy}) + \tfrac{1}{2}(1 - \nu)(v_{,xx} + u_{,xy} + w_{,y}w_{,xx} + w_{,x}w_{,xy}) = 0 \tag{2}$$

$$D\nabla^2\nabla^2 w = q - kw + h\left\{\frac{E}{(1-\nu^2)}\left[u_{,x} + \frac{1}{2}(w_{,x})^2 + \nu\left(v_{,y} + \frac{1}{2}(w_{,y})^2\right)\right]w_{,xx} + \frac{E}{(1-\nu^2)}\left[v_{,y} + \frac{1}{2}(w_{,y})^2 + \nu\left(u_{,x} + \frac{1}{2}(w_{,x})^2\right)\right]w_{,yy} + \frac{E}{(1-\nu^2)} \times (u_{,y} + v_{,x} + w_{,x}w_{,y})w_{,xy}\right\} \tag{3}$$

where ∇ is the Laplacian operator, ν being Poisson's ratio, D the flexural rigidity, E the modulus of elasticity of the plate material; q the intensity of uniformly distributed load, k the foundation modulus; h is the thickness of the plate and the comma notation signifies differentiation.

By adopting the dimensionless ratios

$$R = b/a;\ \xi = x/a;\ \eta = y/b$$

$$U = \frac{ua}{h^2};\ V = \frac{va}{h^2};\ W = \frac{w}{h};\ Q = \frac{qb^4}{Dh};\ K = \frac{kb^4}{D}$$

Table 1 Coefficients w for maximum small deflection at center for various elastic foundation moduli plate aspect ratios $R = b/a$; $W_{\max} = w(b^4q/D)(10^{-2})$

Dimensionless Foundation modulus K	R = 1	R = 1.25	R = 1.50	R = 1.75	R = 2.0
0	1.5625	0.9294	0.5510	0.3355	0.2118
20	1.3017	0.8300	0.5140	0.3212	0.2060
40	1.1143	0.7495	0.4816	0.3080	0.2004
60	0.9730	0.6998	0.4529	0.2958	0.1951
80	0.8627	0.6268	0.4273	0.2846	0.1900
100	0.7741	0.5791	0.4044	0.2741	0.1852
120	0.7013	0.5378	0.3838	0.2643	0.1807
140	0.6405	0.5019	0.3651	0.2552	0.1763
160	0.5889	0.4703	0.3480	0.2467	0.1721
180	0.5446	0.4422	0.3324	0.2387	0.1682
200	0.5060	0.4172	0.3181	0.2312	0.1644

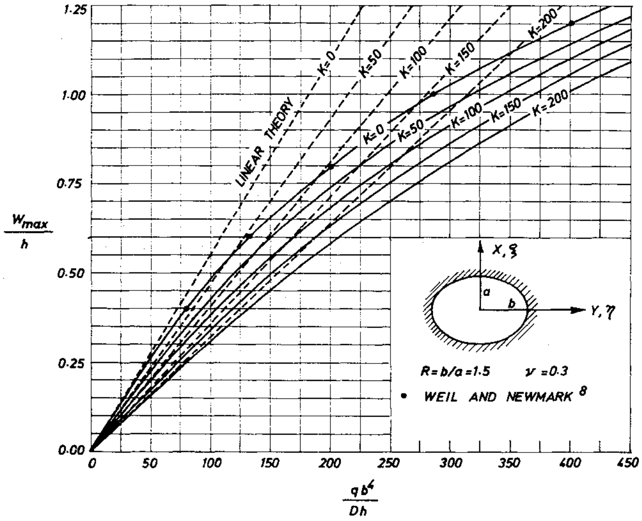

Fig. 1. Variation of center deflections with elastic foundation for aspect ratio $b/a = 1.5$.

Equations (1), (2), and (3) can be written in dimensionless form as

$$2R^3U_{,\xi\xi} + R(1-\nu)U_{,\eta\eta} + R^2(1+\nu)V_{,\xi\eta} + 2R^3W_{,\xi}W_{,\xi\xi} + R(1+\nu)W_{,\eta}W_{,\xi\eta} + R(1-\nu)W_{,\eta\eta}W_{,\xi} = 0 \quad (4)$$

$$2RV_{,\eta\eta} + R^3(1-\nu)V_{,\xi\xi} + R^2(1+\nu)U_{,\xi\eta} + 2W_{,\eta}W_{,\eta\eta} + R^2(1+\nu)W_{,\xi}W_{,\xi\eta} + R^2(1-\nu)W_{,\xi\xi}W_{,\eta} = 0 \quad (5)$$

$$R^4W_{,\xi\xi\xi\xi} + 2R^2W_{,\xi\xi\eta\eta} + W_{,\eta\eta\eta\eta} = (Q - KW) + 12\{[(R^2U_{,\xi}) + \tfrac{1}{2}(RW_{,\xi})^2 + \nu[RV_{,\eta} + \tfrac{1}{2}(W_{,\eta})^2]\}R^2W_{,\xi\xi} + 12R\{V_{,\eta} + \tfrac{1}{2}(W_{,\eta})^2 + \nu[R^2U_{,\xi} + \tfrac{1}{2}(RW_{,\xi})^2]\}W_{,\eta\eta} + 12(1-\nu)[RU_{,\eta} + R^2V_{,\xi} + RW_{,\xi}W_{,\eta}]RW_{,\xi\eta} \quad (6)$$

The inherent boundary conditions that have to be made in a clamped elliptical plate are

$$W = W_{,\xi} = W_{,\eta} = U = V = 0 \text{ at } \xi^2 + \eta^2 = 1 \quad (7)$$

By using W_0 as the perturbation parameter and assuming the solutions of Eqs. (4–6) in the form of a power series, the dimensionless load Q and displacement components U, V, and W can be expanded in ascending powers of the dimensionless center deflection W_0 as

$$Q = \gamma_1W_0 + \gamma_3W_0^3 + \gamma_5W_0^5 + \ldots \quad (8)$$

$$W = w_1(\xi,\eta)W_0 + w_3(\xi,\eta)W_0^3 + w_5(\xi,\eta)W_0^5 + \ldots \quad (9)$$

$$U = s_2(\xi,\eta)W_0^2 + s_4(\xi,\eta)W_0^4 + \ldots \quad (10)$$

$$V = t_2(\xi,\eta)W_0^2 + t_4(\xi,\eta)W_0^4 + \ldots \quad (11)$$

where $\gamma_1, \gamma_3, \gamma_5, \ldots$ are constants relating the center deflection of the plate to the load, and $w_1, w_3, \ldots, s_2, s_4, \ldots, t_2, t_4, \ldots$ are displacement functions satisfying the kinematic boundary conditions of the plate.

From the series for W, Eq. (9), it is evident that, in order that the center deflection be W_0, as defined, it is necessary to require

$$w_1(0,0) = 1 \text{ and } w_3(0,0) = w_5(0,0) = 0 \quad (12)$$

Substitution of Eqs. (5–11) in Eqs. (4–6) and equating, subsequently, successively higher powers of W_0, yield the governing differential equations for displacements corresponding to each stage of the successive approximation process. Next, polynomical expressions for the displacements, satisfying the kinematic boundary of the clamped elliptical plate are assumed and substituted in the resulting governing differential equations. Equating terms with corresponding powers of ξ and η yields sets of linear simultaneous equations from which the undetermined coefficients in the assumed displacement functions can be evaluated. For the sake of brevity, detailed solution procedures and convergence properties of the perturbation method as applied to linear and nonlinear plate problems are not elaborated here but can be found elsewhere[6,7] in the literature.

Numerical results for the small deflection problem obtained from the first-order approximation are shown in Table 1 for various foundation moduli and plate aspect ratios. It can be observed that for any given aspect ratio, the effectiveness of the foundation modulus in reducing the plate deflection decreases as the foundation modulus is increased. Furthermore, it is evident from the numerical results that the influence of the foundation reaction decreases appreciably with an increase in the plate aspect ratio R.

For the limiting case where the foundation modulus K is reduced to zero the first approximation results are in exact agreement with those obtained by Timoshenko.[1] Typical results for the finite deflection problem are shown graphically in Figs. 1 and 2 for aspect ratios $R = 1.5$ and 2, respectively. To exhibit the difference between the linear and nonlinear behavior, results based on the small-deflection theory are shown plotted in the aforementioned figures. From the figures it can be seen that for elliptical plates with no foundation support results obtained by the present finite deflection analysis compare very favorably with those obtained by Weil and Newmark.[8] No data is yet available for finite deflections of elliptical plates on elastic foundations and hence comparison is not possible for deflection values when the foundation modulus is greater than zero. From Figs. 1 and 2 it is evident that the maximum deflection at the center of the plate decreases fairly rapidly with an increase in the value of the foundation modulus. This is to be expected since the effect of the elastic support is to reduce the lateral pressure on the plate. As in the case of the small deflection analysis, the effectiveness of the elastic foundation tends to decrease with an increase in the plate aspect ratio R. Also, the influence of the nonlinear terms on the deflection appears to diminish as the value of the foundation modulus increases. Hence, for any given aspect ratio, the large deflection curves tend to become increasingly linear for relatively high values of the foundation modulus.

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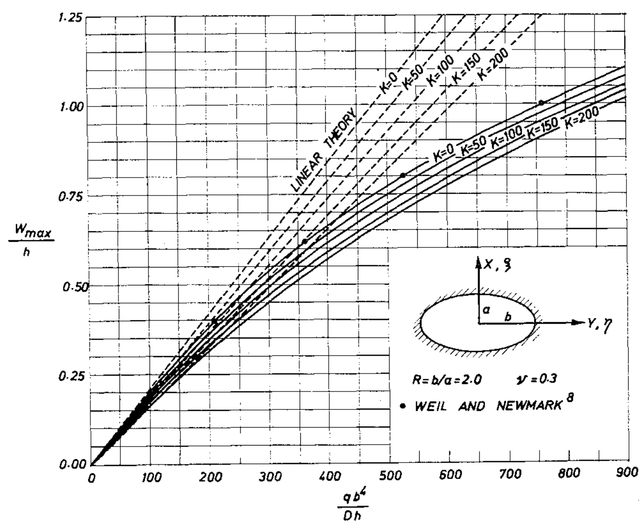

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Use of Control Moment Gyros for the Stabilization of a Spinning Satellite

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I. Introduction

A SET of control moment gyros (CMGs) can be used as the actuator in an attitude control system by being caused to exchange angular momentum with the spacecraft. The advantages of using CMGs for the stabilization of inertially referenced satellites have become well known[1,2] during the development of the attitude control system for the Apollo Telescope Mount of NASA's Skylab Program. This paper is concerned with the feasibility of using CMGs to control a satellite which is spinning relative to a set of inertial coordinates. It is shown that the spin introduces a fundamental difference in the mechanism of opposing applied torques on the spacecraft with the CMGs of an attitude control system. This is related to the general result that the CMGs in the attitude control system of a satellite which is spinning relative to an inertial coordinate system can accumulate bias momentum only about the spin axis of the spacecraft as a result of an applied torque which is constant in spacecraft coordinates. A constant torque applied about a spacecraft axis which is perpendicular to the spin axis results in a periodic momentum requirement on the CMGs. In addition, an example is presented in which the CMG momentum requirement for a spinning satellite is less than that of a similar, nonspinning satellite.

The presentation of these concepts is organized in the following way: Sec. II includes the development of the equations which describe the form of the momentum variation of the CMGs in response to an arbitrary set of external torques; in Sec. III an example of the use of CMGs for stabilizing the attitude of an Earth-pointing satellite is presented; and Sec. IV contains some concluding remarks.

II. A Mathematical Model

The equations of motion relative to a set of coordinates which are fixed in a spacecraft containing CMGs are given by

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\mathbf{I}\boldsymbol{\omega} + \dot{\mathbf{H}} + \tilde{\boldsymbol{\omega}}\mathbf{H} = \mathbf{N} \qquad (1)$$

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where: $\mathbf{I}$ = the 3 × 3 inertia tensor, $\boldsymbol{\omega}$ = a 3 × 1 matrix representing the angular velocity vector of the spacecraft relative to a set of inertial coordinates, $\tilde{\boldsymbol{\omega}}$ = the 3 × 3 skew-symmetric matrix which is isomorphic to the vector cross-product operator, $\mathbf{N}$ = a 3 × 1 matrix representing the total external torque vector acting on the spacecraft, and $\mathbf{H}$ = a 3 × 1 matrix which represents the vector sum of the individual spin angular momenta contributed by each CMG. That is, for a system containing n CMGs with the spin angular momentum of the ith CMG given by $\mathbf{h}_i$,

$$\mathbf{H} = \sum_{i=1}^{n} \mathbf{h}_i \qquad (2)$$

The dots over $\boldsymbol{\omega}$ and $\mathbf{H}$ in Eq. (1) denote the time rate of change of those vectors as measured in a set of coordinates which are fixed in the spacecraft.

Since it is desired to stabilize the spacecraft relative to a spinning reference frame let

$$\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \boldsymbol{\Delta\omega} \qquad (3)$$

where $\boldsymbol{\omega}_0$ is constant. Without loss of generality it is possible to specify $\boldsymbol{\omega}_0' = [w_0\ 0\ 0]$.† That is, a spacecraft coordinate system can be selected with one axis parallel to $\boldsymbol{\omega}_0$. Of course, if this system does not coincide with the spacecraft coordinate system which is implicit in the mounting geometry of a CMG control system, then the results presented below will have to be transformed into the appropriate system in order to determine, say, gimbal excursion. Equations (3) and (1) imply

$$\mathbf{I}\boldsymbol{\Delta}\dot{\boldsymbol{\omega}} + (\tilde{\boldsymbol{\omega}}_0 + \boldsymbol{\Delta}\tilde{\boldsymbol{\omega}})\mathbf{I}(\boldsymbol{\omega}_0 + \boldsymbol{\Delta\omega}) + \dot{\mathbf{H}} + (\tilde{\boldsymbol{\omega}}_0 + \boldsymbol{\Delta}\tilde{\boldsymbol{\omega}})\mathbf{H} = \mathbf{N} \qquad (4)$$

Further, if it is assumed that the CMG control system is capable of maintaining the actual state of the system identically equal to the desired state, then $\boldsymbol{\Delta\omega} = 0$ and (4) can be rewritten as

$$\dot{\mathbf{H}} + \tilde{\boldsymbol{\omega}}_0\mathbf{H} = \mathbf{N} - \boldsymbol{\omega}_0\mathbf{I}\boldsymbol{\omega}_0 = \mathbf{T} \qquad (5)$$

This assumption is certainly justified for the determination of the CMG capacity which will be required to stabilize a particular attitude. The neglected terms are merely perturbations from the solution of Eq. (5).

Equation (5) is a linear, time-invariant system. Writing the Laplace transform of its expanded form yields

$$\begin{aligned} sH_x(s) &= T_x(s) + H_x(0+) \\ sH_y(s) - \omega_0 H_z(s) &= T_y(s) + H_y(0+) \\ \omega_0 H_y(s) + sH_z(s) &= T_z(s) + H_z(0+) \end{aligned} \qquad (6)$$

where the initial time t_0 is taken to be $t_0 = 0+$. Notice that the spin axis momentum (H_x) is merely the integral of the applied torque. This is to be expected since this axis is fixed relative to a set of inertial coordinates. However, the solutions for $H_y(s)$ and $H_z(s)$ are given by

$$\begin{aligned} H_y(s) &= \{s[T_y(s) + H_y(0+)] + \omega_0[T_z(s) + H_z(0+)]\}/(s^2 + \omega_0^2) \\ H_z(s) &= \{-\omega_0[T_y(s) + H_y(0+)] + s[T_z(s) + H_z(0+)]\}/(s^2 + \omega_0^2) \end{aligned} \qquad (7)$$

Three aspects of Eqs. (6) and (7) are particularly important in distinguishing this type of system response from the variation of $\mathbf{H}$ in an inertial attitude: 1) In the spinning coordinates the magnitude of $\mathbf{H}$ is the important quantity rather than $\dot{\mathbf{H}}$. For example, it can be seen from Eq. (6) that any constant torque $\mathbf{T}$ in the $y - z$ plane of the spacecraft can be offset by setting $H_y(t) = T_z/\omega_0$ and $H_z(t) = -T_y/\omega_0$. Thus, there is a fundamental difference in the way that a

† The prime denotes the transpose of the matrix $\boldsymbol{\omega}_0$.